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1 Introduction

This quick reference guide presents a collection of MIP model formulations for Xpress-Optimizer, including standard linearization techniques involving binary variables, the use of more specific modeling objects such as SOS and partial integer variables, and reformulations of logic constraints through indicator constraints.

1.1 Integer Programming entities supported in Xpress

- Binary variables (BV) – decision variables that must take either the value 0 or the value 1, sometimes called 0/1 variables;
• **Integer variables (UI)** – decision variables that must take on integer values. Some upper limit must be specified;

• **Partial integer variables (PI)** – decision variables that must take integer values below a specified limit but can take any value above that limit;

• **Semi-continuous variables (SC)** – decision variables that must take on either the value 0, or any value in a range whose lower and upper limits are specified. SCs help model situations where, if a variable is to be used at all, it has to be at some minimum level;

• **Semi-continuous integer variables (SI)** – decision variables that must take either the value 0, or any integer value in a range whose lower and upper limits are specified;

• **Special ordered sets of type one (SOS1)** – an ordered set of variables of which at most one can take a nonzero value;

• **Special ordered sets of type two (SOS2)** – an ordered set of variables of which at most two can be nonzero, and if two are nonzero, they must be consecutive in their ordering.

**Remarks**

• The solution values of binary and integer variables are real valued, not integer valued.

• At an optimal MIP solution, the actual values of the binary and integer variables will be integer – to within a certain tolerance.

### 1.2 Integer Programming entities in Mosel

**Definition:** integer programming types are defined as unary constraints on previously declared decision variables of type `mpvar`; name the constraints if you want to be able to access/modify them.

```mosel
declarations
d: mpvar
ifmake: array(PRODS,LINES) of mpvar
x: mpvar
end-declarations
d is_binary ! Single binary variable
forall(p in PRODS, l in LINES)
  ifmake(p,l) is_binary ! An array of binaries
ACtr:= x is_integer ! An integer variable
x >= MINVAL ! Lower bound on the variable
x <= MAXVAL ! Upper bound on the variable
MINVAL,MAXVAL: values between -MAX_REAL and MAX_REAL
...
ACtr:= x is_partint 10 ! Change type to partial integer
...
ACtr:= 0 ! Delete constraint
! Equivalently:
ACtr:= x is_continuous ! Change type to continuous
```

**Solving:** with Xpress-Optimizer (Mosel module `mmxprs`) any problem containing integer programming entities is automatically solved as a MIP problem, to solve just the LP relaxation use option `XPRS_TOP` (if following up with MIP search) or `XPRS_LIN` (ignore all MIP information) for **maximize**/**minimize**.

**Accessing the solution:** for obtaining solution values of decision variables and linear expressions use `getsol` (alternative syntax: `.sol`); the MIP problem status is returned by the function `getparam("XPRS_MIPSTATUS")`. 
case getparam("XPRS_MIPSTATUS") of
  XPRS_MIP_NOT_LOADED,
    XPRS_MIP_LP_NOT_OPTIMAL: writeln("Solving not started")
  XPRS_MIP_LP_OPTIMAL: writeln("LP unbounded or infeasible")
  XPRS_MIP_NO_SOL_FOUND,
    XPRS_MIP_INFEAS: writeln("MIP search started, no solution")
  XPRS_MIP_SOLUTION,
    XPRS_MIP_OPTIMAL: writeln("MIP solution: ", , getobjval)
end-case

writeln("x: ", getsol(x))
writeln("d: ", d.sol)

1.3 Integer Programming entities in BCL

The BCL code extracts in this document are formulated for the BCL C++ interface. The other BCL interfaces (C, Java, C#, VB) work similarly, please refer to the Xpress documentation for further detail.

Definition: Integer Programming types are specified when creating decision variables (type XPRBvar); types may be changed with setType.

```cpp
XPRBprob prob("test");
XPRBvar d, ifmake[NP][NL], x;
int p,l;

d = prob.newVar("d", XPRB_BV); // Single binary variable
for (p = 0; p < NP; p++) // An array of binaries
  for (l = 0; l < NL; l++)
    ifmake[p][l] = prob.newVar("ifmake", XPRB_BV);

x = prob.newVar("x", XPRB_UI, MINVAL, MAXVAL); // An integer variable
// MINVAL,MAXVAL: reals between -XPRB_INFINITY and XPRB_INFINITY
... x.setType(XPRB_PI); // Change type to partial integer
x.setLim(10);
... x.setType(XPRB_PL); // Change type to continuous
```

Solving: use option "g" in solve/imin/mimax to solve a problem as a MIP problem (per default only the LP relaxation is solved).

```cpp
prob.minim("g");
```

Accessing the solution: for obtaining solution values of decision variables and linear expressions use getSol; the MIP problem status is returned by getMIPstatus.

```cpp
int mipstatus = prob.getMIPStat();
switch (mipstatus) {
  case XPRB_MIP_NOT_LOADED:
  case XPRB_MIP_LP_NOT_OPTIMAL:
    cout << "Solving not started" << endl;
    break;
  case XPRB_MIP_LP_OPTIMAL:
    cout << "LP unbounded or infeasible" << endl;
    break;
  case XPRB_MIP_NO_SOL_FOUND:
  case XPRB_MIP_INFEAS:
    cout << "MIP search started, no solution" << endl;
    break;
  case XPRB_MIP_SOLUTION:
```
2 Binary variables

Binary decision variables

- take value 0 or 1
- model a discrete decision
  - yes/no
  - on/off
  - open/close
  - build or don’t build
  - strategy A or strategy B

2.1 Logical conditions

Projects A, B, C, D, ... with associated binary variables a, b, c, d, ... which are 1 if we decide to do the project and 0 if we decide not to do the project.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>At most N of A, B, C,...</td>
<td>$a + b + c + \ldots \leq N$</td>
</tr>
<tr>
<td>At least N of A, B, C,...</td>
<td>$a + b + c + \ldots \geq N$</td>
</tr>
<tr>
<td>Exactly N of A, B, C,...</td>
<td>$a + b + c + \ldots = N$</td>
</tr>
<tr>
<td>If A then B</td>
<td>$b \geq a$</td>
</tr>
<tr>
<td>Not B</td>
<td>$b = 1 - b$</td>
</tr>
<tr>
<td>If A then not B</td>
<td>$a + b \leq 1$</td>
</tr>
<tr>
<td>If not A then B</td>
<td>$a + b \geq 1$</td>
</tr>
<tr>
<td>If A then B, and if B then A</td>
<td>$a = b$</td>
</tr>
</tbody>
</table>
| If A then B and C; A only if B and C       | $b \geq a$ and $c \geq a$ or alternatively: $a \leq (b + c) / 2$
| If A then B or C                           | $b + c \geq a$                       |
| If B or C then A                           | $a \geq b$ and $a \geq c$ or alternatively: $a \geq \frac{1}{2} \cdot (b + c)$ |
| If B and C then A                          | $a \geq b + c - 1$                   |
| If two or more of B, C, D or E then A      | $a \geq \frac{1}{3} \cdot (b + c + d + e - 1)$ |
| If M or more of N projects (B, C, D, ...) then A | $a \geq \frac{b + c + d + \ldots - M + 1}{N - M + 1}$ |

2.2 Minimum values

$y = \min\{x_1, x_2\}$ for two continuous variables $x_1, x_2$

- Must know lower and upper bounds
MIP formulations

\[ L_{1} \leq x_{1} \leq U_{1} \quad [1.1] \]
\[ L_{2} \leq x_{2} \leq U_{2} \quad [1.2] \]

- Introduce binary variables \( d_{1}, d_{2} \) to mean
  \( d_{i} = 1 \) if \( x_{i} \) is the minimum value;
  \( 0 \) otherwise

- MIP formulation:
  \[ y \leq x_{1} \quad [2.1] \]
  \[ y \leq x_{2} \quad [2.2] \]
  \[ y \geq x_{1} - (U_{1} - L_{\text{min}})(1 - d_{1}) \quad [3.1] \]
  \[ y \geq x_{2} - (U_{2} - L_{\text{min}})(1 - d_{2}) \quad [3.2] \]
  \[ d_{1} + d_{2} = 1 \quad [4] \]

- Generalization to \( y = \min\{x_{1}, x_{2}, \ldots, x_{n}\} \)
  \[ L_{i} \leq x_{i} \leq U_{i} \quad [1.i] \]
  \[ y \leq x_{i} \quad [2.i] \]
  \[ y \geq x_{i} - (U_{i} - L_{\text{min}})(1 - d_{i}) \quad [3.i] \]
  \[ \sum_{i} d_{i} = 1 \quad [4] \]

### 2.3 Maximum values

\( y = \max\{x_{1}, x_{2}, \ldots, x_{n}\} \) for continuous variables \( x_{1}, \ldots, x_{n} \)

- Must know lower and upper bounds
  \[ L_{i} \leq x_{i} \leq U_{i} \quad [1.i] \]

- Introduce binary variables \( d_{1}, \ldots, d_{n} \)
  \( d_{i} = 1 \) if \( x_{i} \) is the minimum value, \( 0 \) otherwise

- MIP formulation
  \[ L_{i} \leq x_{i} \leq U_{i} \quad [1.i] \]
  \[ y \geq x_{i} \quad [2.i] \]
  \[ y \leq x_{i} + (U_{\text{max}} - L_{i})(1 - d_{i}) \quad [3.i] \]
  \[ \sum_{i} d_{i} = 1 \quad [4] \]

### 2.4 Absolute values

\( y = |x_{1} - x_{2}| \) for two variables \( x_{1}, x_{2} \) with \( 0 \leq x_{i} \leq U \)

- Introduce binary variables \( d_{1}, d_{2} \) to mean
  \( d_{1} = 1 \) if \( x_{1} - x_{2} \) is the positive value
  \( d_{2} = 1 \) if \( x_{2} - x_{1} \) is the positive value
2.5 Logical AND

\[ d = \min\{d_1, d_2\} \text{ for two binary variables } d_1, d_2, \text{ or equivalently} \]

\[ d = d_1 \cdot d_2 \text{ (see Section 2.8), or} \]

\[ d = d_1 \text{ AND } d_2 \text{ as a logical expression} \]

2.6 Logical OR

\[ d = \max\{d_1, d_2\} \text{ for two binary variables } d_1, d_2, \text{ or} \]

\[ d = d_1 \text{ OR } d_2 \text{ as a logical expression} \]
2.7 Logical NOT

\[ d = \text{NOT} \, d_1 \] for one binary variable \( d_1 \)

- IP formulation

\[ d = 1 - d_1 \]

2.8 Product values

\[ y = x \cdot d \] for one continuous variable \( x \), one binary variable \( d \)

- Must know lower and upper bounds

\[ L \leq x \leq U \]

- MIP formulation:

\[
\begin{align*}
Ld & \leq y \leq Ud \quad [1] \\
L(1 - d) & \leq x - y \leq U(1 - d) \quad [2]
\end{align*}
\]

Product of two binaries: \( d_3 = d_1 \cdot d_2 \)

- MIP formulation:

\[
\begin{align*}
d_3 & \leq d_1 \\
d_3 & \leq d_2 \\
d_3 & \geq d_1 + d_2 - 1
\end{align*}
\]

2.9 Disjunctions

Either \( 5 \leq x \leq 10 \) or \( 80 \leq x \leq 100 \)

- Introduce a new binary variable:

\( \text{ifupper} \): 0 if \( 5 \leq x \leq 10 \); 1 if \( 80 \leq x \leq 100 \)

- MIP formulation:

\[
\begin{align*}
x & \leq 10 + (100 - 10) \cdot \text{ifupper} \quad [1] \\
x & \geq 5 + (80 - 5) \cdot \text{ifupper} \quad [2]
\end{align*}
\]

- Generalization to Either \( L_1 \leq \sum_i A_i x_i \leq U_1 \) or \( L_2 \leq \sum_i A_i x_i \leq U_2 \) (with \( U_1 \leq L_2 \))

\[
\begin{align*}
\sum_i A_i x_i & \leq U_1 + (U_2 - U_1) \cdot \text{ifupper} \quad [1] \\
\sum_i A_i x_i & \geq L_1 + (L_2 - L_1) \cdot \text{ifupper} \quad [2]
\end{align*}
\]
2.10 Minimum activity level

Continuous production rate \( \text{make} \) that may be 0 (the plant is not operating) or between allowed production limits \( \text{MAKEMIN} \) and \( \text{MAKEMAX} \)

- Introduce a binary variable \( \text{ifmake} \) to mean
  \[ \text{ifmake} : \begin{align*}
  0 & \text{ if plant is shut} \\
  1 & \text{ plant is open}
  \end{align*} \]

MIP formulation:

\[
\begin{align*}
\text{make} & \geq \text{MAKEMIN} \cdot \text{ifmake} \quad [1] \\
\text{make} & \leq \text{MAKEMAX} \cdot \text{ifmake} \quad [2]
\end{align*}
\]

Note: see Section 3.5 for an alternative formulation using semi-continuous variables

- The \( \text{ifmake} \) binary variable also allows us to model fixed costs
  - \( \text{FCOST} \): fixed production cost
  - \( \text{VCOST} \): variable production cost

MIP formulation:

\[
\begin{align*}
\text{cost} & = \text{FCOST} \cdot \text{ifmake} + \text{VCOST} \cdot \text{make} \quad [3] \\
\text{make} & \geq \text{MAKEMIN} \cdot \text{ifmake} \quad [1] \\
\text{make} & \leq \text{MAKEMAX} \cdot \text{ifmake} \quad [2]
\end{align*}
\]

3 MIP formulations using other entities

In principle, all you need in building MIP models are continuous variables and binary variables. But it is convenient to extend the set of modeling entities to embrace objects that frequently occur in practice.

**Integer decision variables**

- values 0, 1, 2, ... up to small upper bound
- model discrete quantities
- try to use *partial integer variables* instead of integer variables with a very large upper bound

**Semi-continuous variable**

- may be zero, or any value between the intermediate bound and the upper bound
- *Semi-continuous integer variables* also available: may be zero, or any integer value between the intermediate bound and the upper bound

**Special ordered sets**

- set of decision variables
• each variable has a different ordering value, which orders the set
• Special ordered sets of type 1 (SOS1): at most one variable may be non-zero
• Special ordered sets of type 2 (SOS2): at most two variables may be non-zero; the non-zero variables must be adjacent in ordering

3.1 Batch sizes

Must deliver in batches of 10, 20, 30, ...

• Decision variables
  \[ nship \quad \text{number of batches delivered: integer} \]
  \[ ship \quad \text{quantity delivered: continuous} \]

• Constraint formulation
  \[ ship = 10 \cdot nship \]

3.2 Ordered alternatives

Suppose you have \( N \) possible investments of which at most one can be selected. The capital cost is \( CAP_i \) and the expected return is \( RET_i \).

• Often use binary variables to choose between alternatives. However, SOS1 are more efficient to choose between a set of graded (ordered) alternatives.
• Define a variable \( d_i \) to represent the decision, \( d_i = 1 \) if investment \( i \) is picked
• Binary variable (standard) formulation
  \[ d_i : \text{binary variables} \]
  \[
  \begin{align*}
  \text{Maximize: } ret &= \sum_i RET_i d_i \\
  \sum_i d_i &\leq 1 \\
  \sum_i CAP_i d_i &\leq \text{MAXCAP}
  \end{align*}
  \]
• SOS1 formulation
  \[
  \{d_i; \text{ordering value } CAP_i\} : \text{SOS1} \\
  \begin{align*}
  \text{Maximize: } ret &= \sum_i RET_i d_i \\
  \sum_i d_i &\leq 1 \\
  \sum_i CAP_i d_i &\leq \text{MAXCAP}
  \end{align*}
  \]
• Special ordered sets in Mosel
  – special ordered sets are a special type of linear constraint
  – the set includes all variables in the constraint
  – the coefficient of a variable is used as the ordering value (i.e., each value must be unique)
declarations
I=1..4
d: array(I) of mpvar
CAP: array(I) of real
My_Set, Ref_row: linctr
end-declarations

My_Set:= sum(i in I) CAP(i)*d(i) is sos1

or alternatively (must be used if a coefficient is 0):

Ref_row:= sum(i in I) CAP(i)*d(i)
makesos1(My_Set, union(i in I) d(i), Ref_row)

• Special ordered sets in BCL
  – a special ordered set is an object of type XPRBsos
  – the set includes all variables from the specified linear expression or constraint that have a coefficient different from 0
  – the coefficient of a variable is used as the ordering value (i.e., each value must be unique)

XPRBprob prob("testsos");
XPRBvar d[I];
XPRBexpr le;
XPRBsos My_Set;
double CAP[I];
int i;

for(i=0; i<I; i++) d[i] = prob.newVar("d");
for(i=0; i<I; i++) le += CAP[i]*d[i];
My_Set = prob.newSos("My_Set", XPRB_S1, le);

3.3 Price breaks

All items discount: when buying a certain number of items we get discounts on all items that we buy if the quantity we buy lies in certain price bands.

less than $B_1$  
$\geq B_1$ and $< B_2$  
$\geq B_2$ and $< B_3$

• Define binary variables $b_i$ (i=1,2,3), where $b_i$ is 1 if we pay a unit cost of $COST_i$.
• Real decision variables $x_i$ represent the number of items bought at price $COST_i$. 
The total cost is given by \( x = \sum_i x_i \)

**MIP formulation:**

\[
\begin{align*}
\sum_i b_i &= 1 \\
x_1 &\leq B_1 \cdot b_1 \\
B_{i-1} \cdot b_i &\leq x_i \leq B_i \cdot b_i \text{ for } i = 2, 3
\end{align*}
\]

where the variables \( b_i \) are either defined as binaries, or they form a Special Ordered Set of type 1 (SOS1), where the order is given by the values of the breakpoints \( B_i \).

**Incremental pricebreaks:** when buying a certain number of items we get discounts incrementally. The unit cost for items between 0 and \( B_1 \) is \( C_1 \), items between \( B_1 \) and \( B_2 \) cost \( C_2 \) each, etc.

![Graph showing incremental pricebreaks](image)

**Formulation with Special Ordered Sets of type 2 (SOS2):**

- Associate real valued decision variables \( w_i \) \( (i = 0, 1, 2, 3) \) with the quantity break points \( B_0 = 0, B_1, B_2 \) and \( B_3 \).
- Cost break points \( CBP_i \) (=total cost of buying quantity \( B_i \)):
  \[
  \begin{align*}
  CBP_0 &= 0 \\
  CBP_i &= CBP_{i-1} + C_i \cdot (B_i - B_{i-1}) \text{ for } i = 1, 2, 3
  \end{align*}
  \]
- **Constraint formulation:**
  \[
  \begin{align*}
  \sum_i w_i &= 1 \\
  TotalCost &= \sum_i CBP_i \cdot w_i \\
  x &= \sum_i B_i \cdot w_i
  \end{align*}
  \]

where the \( w_i \) form a SOS2 with reference row coefficients given by the coefficients in the definition of the total amount \( x \).

For a solution to be valid, at most two of the \( w_i \) can be non-zero, and if there are two non-zero they must be contiguous, thus defining one of the line segments.

- **Implementation with Mosel** (\textit{is\_sos2} cannot be used here due to the 0-valued coefficient of \( w_0 \)):
  \[
  \begin{align*}
  \text{Defx := x = sum\{i in 1..3\} B(i) \cdot w(i)} \\
  \text{makesos2(My\_Set, union\{i in 0..3\} w(i), Defx)
  \end{align*}
  \]

**Formulation using binaries:**

\[
\text{Defx := x = sum\{i in 1..3\} B(i) \cdot w(i), makesos2(My\_Set, union\{i in 0..3\} w(i), Defx)
}
MIP formulations

- Define binary variables \( b_i \) (\( i=1,2,3 \)), where \( b_i \) is 1 if we have bought any items at a unit cost of \( \text{COST}_i \).
- Real decision variables \( x_i \) (\( i=1,..3 \)) for the number of items bought at price \( \text{COST}_i \).
- Total amount bought: \( x = \sum x_i \)
- Constraint formulation:
  \[
  (B_i - B_{i-1}) \cdot b_{i+1} \leq x_i \leq (B_i - B_{i-1}) \cdot b_i \text{ for } i = 1, 2 \\
  x_3 \leq (B_3 - B_2) \cdot b_3 \\
  b_1 \geq b_2 \geq b_3
  \]

### 3.4 Non-linear functions

Can model non-linear functions in the same way as incremental pricebreaks

- approximate the non-linear function with a piecewise linear function
- use an SOS2 to model the piecewise linear function

#### Non-linear function in a single variable

- \( x \)-coordinates of the points: \( R_1, \ldots, R_4 \)
- \( y \)-coordinates \( F_1, \ldots, F_4 \). So point 1 is \((R_1, F_1)\) etc.
- Let weights (decision variables) associated with point \( i \) be \( w_i \) (\( i=1,...,4 \))
- Form convex combinations of the points using weights \( w_i \) to get a combination point \((x,y)\):
  \[
  x = \sum w_i \cdot R_i \\
  y = \sum w_i \cdot F_i \\
  \sum w_i = 1
  \]

where the variables \( w_i \) form an SOS2 set with ordering coefficients defined by values \( R_i \).
MIP formulations

• Mosel implementation:

```mosel
declarations
I=1..4
x,y: mpvar
w: array(I) of mpvar
R,F: array(I) of real
end-declarations

! ...assign values to arrays R and F...

! Define the SOS-2 with "reference row" coefficients from R
Defx:= sum(i in I) R(i)*w(i) is_sos2
sum(i in I) w(i) = 1

! The variable and the corresponding function value we want to approximate
x = Defx
y = sum(i in I) F(i)*w(i)
```

• BCL implementation:

```cbl
XPRBprob prob("testsos");
XPRBvar x, y, w[I];
XPRBexpr Defx, le, ly;
double R[I], F[I];
int i;

// ...assign values to arrays R and F...

// Create the decision variables
x = prob.newVar("x"); y = prob.newVar("y");
for(i=0; i<I; i++) w[i] = prob.newVar("w");

// Define the SOS-2 with "reference row" coefficients from R
for(i=0; i<I; i++) Defx += R[i]*w[i];
prob.newSos("Defx", XPRB_S2, Defx);
for(i=0; i<I; i++) le += w[i];
prob.newCtr("One", le == 1);

// The variable and the corresponding function value we want to approximate
prob.newCtr("Eqx", x == Defx);
for(i=0; i<I; i++) ly += F[i]*w[i];
prob.newCtr("Eqy", y == ly);
```

Non-linear function in two variables

Interpolation of a function \( f \) in two variables: approximate \( f \) at a point \( P \) by the corners \( C \) of the enclosing square of a rectangular grid (NB: the representation of \( P=(x,y) \) by the four points \( C \) obviously means a fair amount of degeneracy).
- x-coordinates of grid points: \( X_1, ..., X_n \)
- y-coordinates of grid points: \( Y_1, ..., Y_m \). So grid points are \((X_i, Y_j)\).

- Function evaluation at grid points: \( F_{XY_{11}}, ..., F_{XY_{nm}} \)
- Define weights (decision variables) associated with x and y coordinates, \( wx_i \) respectively \( wy_j \), and for each grid point \((X(i), Y(j))\) define a variable \( wxy_{ij} \)
- Form convex combinations of the points using the weights to get a combination point \((x, y)\) and the corresponding function approximation:

\[
\begin{align*}
x &= \sum_i wx_i \cdot X_i \\
y &= \sum_j wy_j \cdot Y_j \\
f &= \sum_{ij} wxy_{ij} \cdot F_{XY_{ij}} \\
\forall i &= 1, \ldots, n : \sum_j wxy_{ij} = wx_i \\
\forall j &= 1, \ldots, m : \sum_i wxy_{ij} = wy_j \\
\sum_i wx_i &= 1 \\
\sum_j wy_j &= 1
\end{align*}
\]

where the variables \( wx_i \) form an SOS2 set with ordering coefficients defined by values \( X_i \), and the variables \( wy_j \) are a second SOS2 set with coordinate values \( Y_j \) as ordering coefficients.

- Mosel implementation:

```mosel
declarations
RX,RY:range
X: array(RX) of real ! x coordinate values of grid points
Y: array(RY) of real ! y coordinate values of grid points
FXY: array(RX,RY) of real ! Function evaluation at grid points
end-declarations

! ... initialize data

declarations
wx: array(RX) of mpvar ! Weight on x coordinate
wy: array(RY) of mpvar ! Weight on y coordinate
wxy: array(RX,RY) of mpvar ! Weight on (x,y) coordinates
x,y,f: mpvar
end-declarations

! Definition of SOS (assuming coordinate values <>0)
sum(i in RX) X(i)*wx(i) is_sos2
sum(j in RY) Y(j)*wy(j) is_sos2

! Constraints
forall(i in RX) sum(j in RY) wxy(i,j) = wx(i)
forall(j in RY) sum(i in RX) wxy(i,j) = wy(j)
sum(i in RX) wx(i) = 1
sum(j in RY) wy(j) = 1

! Then x, y and f can be calculated using
x = sum(i in RX) X(i)*wx(i)
y = sum(j in RY) Y(j)*wy(j)
f = sum(i in RX, j in RY) FXY(i,j)*wxy(i,j)

! f can take negative or positive values (unbounded variable)
f is_free
```

MIP formulations using other entities

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MIP formulations

- BCL implementation:

```cpp
XPRBprob prob("testsos");
XPRBvar x, y, f;
XPRBvar wx[NX], wy[NY], wxy[NX][NY];  // Weights on coordinates
XPRBexpr Defx, Defy, le, lexy, lx, ly;
double DX[NX], DY[NY];
double FXY[NX][NY];
int i,j;

// ... initialize data arrays DX, DY, FXY

// Create the decision variables
x = prob.newVar("x"); y = prob.newVar("y");
f = prob.newVar("f", XPRB_PL, -XPRB_INFINITY, XPRB_INFINITY);  // Unbounded variable
for(i=0; i<NX; i++) wx[i] = prob.newVar("wx");
for(j=0; j<NY; j++) wy[j] = prob.newVar("wy");
for(i=0; i<NX; i++)
    for(j=0; j<NY; j++) wxy[i][j] = prob.newVar("wxy");

// Definition of SOS
for(i=0; i<NX; i++) Defx += X[i]*wx[i];
prob.newSos("Defx", XPRB_S2, Defx);
for(j=0; j<NY; j++) Defy += Y[j]*wy[j];
prob.newSos("Defy", XPRB_S2, Defy);

// Constraints
for(i=0; i<NX; i++){
    le = 0;
    for(j=0; j<NY; j++) le += wxy[i][j];
    prob.newCtr("Sumx", le == wx[i]);
}
for(j=0; j<NY; j++){
    le = 0;
    for(i=0; i<NX; i++) le += wxy[i][j];
    prob.newCtr("Sumy", le == wy[j]);
}
for(i=0; i<NX; i++) lx += wx[i];
prob.newCtr("Convx", lx == 1);
for(j=0; j<NY; j++) ly += wy[j];
prob.newCtr("Convy", ly == 1);

// Calculate x, y and the corresponding function value f we want to approximate
prob.newCtr("Eqx", x == Defx);
prob.newCtr("Eqy", y == Defy);
for(i=0; i<NX; i++)
    for(j=0; j<NY; j++) lexy += FXY[i][j]*wxy[i][j];
prob.newCtr("Eqy", f == lexy);
```

3.5 Minimum activity level

Continuous production rate `make`. May be 0 (the plant is not operating) or between allowed production limits `MAKEMIN` and `MAKEMAX`

- Can impose using a *semi-continuous variable*: may be zero, or any value between the intermediate bound and the upper bound
- Mosel:

```mosel
make is_semcont MAKEMIN
make <= MAKEMAX
```
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• BCL:

```perl
make = prob.newVar("make", XPRB_SC, 0, MAKEMAX);
make.setLim(MAKEMIN);
```

• Semi-continuous variables are slightly more efficient than the alternative binary variable formulation that we saw before. But if you incur fixed costs on any non-zero activity, you must use the binary variable formulation (see Section 2.10).

3.6 Partial integer variables

• In general, try to keep the upper bound on integer variables as small as possible. This reduces the number of possible integer values, and so reduces the time to solve the problem.

• Sometimes this is not possible – a variable has a large upper bound and must take integer values.
  ⇒ Try to use partial integer variables instead of integer variables with a very large upper bound: takes integer values for small values, where it is important to be precise, but takes real values for larger values, where it is OK to round the value afterwards.

• For example, it may be important to clarify whether the value is 0, 1, 2, ..., 10, but above 10 it is OK to get a real value and round it.

• Mosel:

```mosel
x is_partint 10 ! x is integer valued from 0 to 10
x <= 20 ! x takes real values from 10 to 20
```

• BCL:

```perl
x = prob.newVar("x", XPRB_PI, 0, 20);
x.setLim(10);
```

4 Indicator constraints

**Indicator constraints**

• associate a binary variable \( b \) with a linear constraint \( C \)

• model an implication:
  ‘if \( b = 1 \) then \( C \)’, in symbols: \( b \rightarrow C \), or
  ‘if \( b = 0 \) then \( C \)’, in symbols: \( \overline{b} \rightarrow C \)
  (the constraint \( C \) is active only if the condition is true)

• use indicator constraints for the composition of logic expressions

**Indicator constraints in Mosel:** for the definition of indicator constraints (function `indicator` of module `mmxprs`) you need a binary variable (type `mpvar`) and a linear inequality constraint (type `linctr`). You also have to specify the type of the implication (1 for \( b \rightarrow C \) and -1 for \( \overline{b} \rightarrow C \)). The subroutine `indicator` returns a new constraint of type `logctr` that can be used in the composition of other logic expressions (see Section 4.2 below).
uses "mmxprs"

declarations
R=1..10
C: array(range) of linctr
L: array(range) of logctr
x: array(R) of mpvar
b: array(R) of mpvar
end-declarations

forall (i in R) b(i) is_binary ! Variables for indicator constraints
C(2):= x(2)<=5

! Define 2 indicator constraints
L(1):= indicator(1, b(1), x(1)+x(2)>=12) ! b(1)=1 -> x(1)+x(2)>=12
indicator(-1, b(2), C(2)) ! b(2)=0 -> x(2)<=5

C(2):=0 ! Delete auxiliary constraint definition

Indicator constraints in BCL: an indicator constraint is defined by associating a binary decision variable (XPRBvar) and an integer flag (1 for \( b \rightarrow C \) and \(-1 \) for \( \bar{b} \rightarrow C \)) with a linear inequality or range constraint (XPRBctr). By defining an indicator constraint (function XPRBsetIndicator or method XPRBctr.setIndicator() depending on the host language) the type of the constraint itself gets changed; it can be reset to ‘standard constraint’ by calling the setIndicator function with flag value 0.

XPRBprob prob("testind");
XPRBvar x[N], b[N];
XPRBctr IndCtr[N];
int i;

// Create the decision variables
for(i=0;i<N;i++) x[i] = prob.newVar("x", XPRB_PL); // Continuous variables
for(i=0;i<N;i++) b[i] = prob.newVar("b", XPRB_BV); // Indicator variables

// Define 2 linear inequality constraints
IndCtr[0] = prob.newCtr("L1", x[0]+x[1]>=12);
IndCtr[1] = prob.newCtr("L2", x[1]<=5);

// Turn the 2 constraints into indicator constraints
IndCtr[0].setIndicator(1, b[0]); // b(0)=1 -> x(0)+x(1)>=12
IndCtr[1].setIndicator(-1, b[1]); // b(1)=0 -> x(1)<=5

4.1 Inverse implication

\( b \leftarrow ax \geq b \)

• Model as

\( \bar{b} \leftarrow ax \leq b - m \)

where \( m \) is a sufficiently small value (slightly larger than the feasibility tolerance)

\( b \leftarrow ax \leq b \)

• Model as

\( \bar{b} \leftarrow ax \geq b + m \)
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\[ b \leftarrow ax = b \]

- Model as

\[ b \rightleftharpoons b_1 + b_2 = 1 \]
\[ b_1 \rightarrow ax \geq b + m \]
\[ b_2 \rightarrow ax \leq b - m \]

4.2 Logic constructs

Mosel provides the type `logctr` for defining and working with logic constraints in MIP models. The implementation of these constraints is based on indicator constraints. Logic constraints are composed with linear constraints using the operations `and`, `or`, `xor`, `implies`, and `not` as shown in the following example. Mosel models using logic constraints must include the package `advmod` instead of the Optimizer library `mmxprs`.

uses "advmod"

! **** 'implies', 'not', and 'and' ****

declarations
R = 1..3
C: array(range) of linctr
x: array(R) of mpvar
end-declarations

C(1):= x(1) >= 10
C(2):= x(2) <= 5
C(3):= x(1) + x(2) >= 12

implies(C(1), C(3) and not C(2))
forall() in 1..3 C(j):= 0 ! Delete the auxiliary constraints

! Same as:
implies(x(1) >= 10, x(1) + x(2) >= 12 and not x(2) <= 5)

! **** 'or' and 'xor' ****

declarations
p: array(1..6) of mpvar
end-declarations

forall(1 in 1..6) p(1) is_binary

! Choose at least one of projects 1, 2, 3 (option A)
! or at least two of projects 2, 4, 5, 6 (option B)
\[ p(1) + p(2) + x(3) = 1 \text{ or } p(2) + p(4) + p(5) + p(6) = 2 \]

! Choose either option A or option B, but not both
\[ xor(p(1) + p(2) + p(3) = 1, x(2) + p(4) + p(5) + p(6) = 2) \]

These logic constructs, particularly the logic `or`, can be used for the formulation of minimum or maximum values of a set of variables and also for absolute values:

- **Minimum values**: \( y = \min\{x_1, x_2, \ldots, x_n\} \) for continuous variables \( x_1, \ldots, x_n \)
  
  - Logic formulation:
    \[ y \leq x_i \forall i = 1, \ldots, n \]
    \[ y \geq x_1 \text{ or } \ldots \text{ or } y \geq x_n \]
• Maximum values: $y = \max\{x_1, x_2, \ldots, x_n\}$ for continuous variables $x_1, \ldots, x_n$
  - Logic formulation:
    
    $y \geq x_i \quad \forall i = 1, \ldots, n$
    
    $y \leq x_1 \text{ or } \ldots \text{ or } y \leq x_n$

• Absolute values: $y = |x_1 - x_2|$ for two variables $x_1, x_2$
  - Modeling $y = |x_1 - x_2|$ is equivalent to $y = \max\{x_1 - x_2, x_2 - x_1\}$
  - Logic formulation:
    
    $y \geq x_1 - x_2$
    
    $y \geq x_2 - x_1$
    
    $y \leq x_1 - x_2 \text{ or } y \leq x_2 - x_1$

• Example implementation with Mosel:

```mosel
declarations
  x: array(1..2) of mpvar
  y, u, v: mpvar
  C1, C2: linctr
  C3: logctr
end-declarations

! Formulation of $y = \min\{x(1), x(2)\}$
C1:= y <= x(1)
C2:= y <= x(2)
C3:= y >= x(1) or y >= x(2)

! Formulation of $u = \max\{x(1), x(2)\}$
C1:= u >= x(1)
C2:= u >= x(2)
C3:= u <= x(1) or u <= x(2)

! Formulation of $v = |x(1) - x(2)|$
C1:= v >= x(1) - x(2)
C2:= v >= x(2) - x(1)
C3:= v <= x(1) - x(2) or v <= x(2) - x(1)
```